Throughout, all variables represent integers unless otherwise specified.

For some of the problems on this assignment, it is helpful to observe that \( d \mid n \) if and only if \( (n/d)\mid n \).

1. Solve each congruence.

   (a) \( x^2 + 7x + 10 \equiv 0 \pmod{11} \).

   \textbf{Solution:} \( x^2 + 7x + 10 = (x + 2)(x + 5) \), so \( x = -2, -5 \) are solutions. These are 9, 6 \( \pmod{11} \).

   (b) \( 3x^2 + 9x + 7 \equiv 0 \pmod{13} \)

   \textbf{Solution:}
   
   \[ 3x^2 + 9x + 7 \equiv 0 \pmod{13} \]
   
   \[ 9x^2 + 27x + 21 \equiv 0 \pmod{13} \]
   
   \[ 36x^2 + 108x + 84 \equiv 0 \pmod{13} \]
   
   \[ (6x + 9)^2 + 3 \equiv 0 \pmod{13} \]
   
   \[ (6x + 9)^2 \equiv 10 \pmod{13} \]
   
   \[ 6x + 9 \equiv \pm 6 \pmod{13} \].

   Now 6x \equiv -3 \pmod{13} or 6x \equiv -15 \pmod{13}. 6x \equiv 10 \pmod{13} or 6x \equiv 11 \pmod{13}. Thus x \equiv 6 \pmod{13} or x \equiv 4 \pmod{13}.

   (c) \textbf{TURN IN:} \( 5x^2 + 3x + 4 \equiv 0 \pmod{37} \). [Don’t just guess and check – show me a method!]

2. Prove that \( 6x^2 + 5x + 1 \equiv 0 \pmod{p} \) has a solution for all primes \( p \) but has no solutions in the integers.

\textbf{Solution:}

Since \( 6x^2 + 5x + 1 = (3x+1)(2x+1) \), we need only solve a linear congruence \( (3x+1 \equiv 0 \pmod{p} \) or \( 2x + 1 \equiv 0 \pmod{p} \)) to solve the quadratic congruence. These both have solutions for all primes \( p \neq 2, 3 \), and the original congruence is just \( 5x + 1 \equiv 0 \pmod{p} \) for \( p = 2, 3 \). On the other hand, the real solutions are \( x = -\frac{1}{2} \) and \( x = -\frac{1}{3} \), neither of which is an integer.

3. (a) Show that 7 and 18 are solutions of \( x^2 \equiv -1 \pmod{5^2} \). [There is a theorem that tells us how many solutions there are; in this case, there are only these 2.]

\textbf{Solution:} Just check.

(b) Use (a) to find solutions of \( x^2 \equiv -1 \pmod{5^3} \).

\textbf{Solution:} Let \( x_0 = 7 \). We need to find \( q \) such that \( 7^2 - 5^2q = -1 \). This is satisfied for \( q = 2 \). Now we require \( y \) such that \( 2 \cdot 7y \equiv -2 \pmod{5} \); \( y = 2 \) is a solution. Let \( x_1 = 7 + 2(5^2) = 57 \). Verify that this is a solution.

4. Solve each congruence.
(a) \( x^2 \equiv 7 \pmod{3^3} \)

**Solution:**
Since \( 4^2 \equiv 7 \pmod{9} \), we have a solution \( x_0 = 4 \pmod{3} \). Write \( 4^2 - 3^2(1) = 7 \) and then solve \( 2(4)y = -1 \pmod{3} \) for \( y \). We have \( y = 1 \) as a solution. Now let \( x_1 = 4 + 1(3^2) = 13 \). Verify that \( x_1 = 13 \) is a solution.

(b) **TURN IN:** \( x^2 \equiv 14 \pmod{5^3} \)

(c) \( x^2 \equiv 2 \pmod{7^3} \)

**Solution:** Notice \( x_0 = 10 \) is a solution mod \( 7^2 \): \( 10^2 = 100 \equiv 2 \pmod{7^2} \). Write \( 10^2 - 7^2(2) = 2 \) and then solve \( 2(10)y \equiv -2 \pmod{7} \). We find \( y = 2 \), giving \( x_1 = 10 + 2(7^2) = 108 \). Verify that this is a solution.

(d) \( x^2 + 5x + 6 \equiv 0 \pmod{5^3} \)

**Solution:** \( x^2 + 5x + 6 = (x + 2)(x + 3) \), so \( x = -2, x = -3 \) are solutions.

(e) \( x^2 + x + 3 \equiv 0 \pmod{3^3} \)

**Solution:** Recall that completing the square to solve \( ax^2 + bx + c \equiv 0 \pmod{q} \) gives us \( (2ax + b)^2 \equiv b^2 - 4ac \pmod{q} \). In our case, this means \( (2x + 1)^2 \equiv -11 \equiv 16 \pmod{3^3} \). Accordingly, we will first solve \( y^2 \equiv 16 \pmod{3^3} \). We see that \( y = 4 \) is a solution, so we need to solve \( 2x + 1 \equiv 4 \pmod{27} \), which gives \( 2x \equiv 3 \pmod{27} \). Thus \( 14(2x) \equiv 14(3) \pmod{27} \), so \( x \equiv 15 \). The other solution comes from \( y = -4 \), or \( 2x + 1 \equiv -4 \pmod{27} \), giving \( x = 11 \).

5. Compute each valuation.

(a) \( \left[ \frac{22}{7} \right]_7 \)

(b) \( \left[ \frac{355}{113} \right]_{11} \)

(c) \( \left[ \frac{243}{85} \right]_3 \)

**Solutions:** \( \left[ \frac{22}{7} \right]_7 = 7^1 \), \( \left[ \frac{355}{113} \right]_{11} = 1 \), and \( \left[ \frac{243}{85} \right]_3 = 1 \).

6. **TURN IN:** Consider the sequence 1, 2, 4, 8, 16, \ldots, \( 2^n \), \ldots. What are the 2-adic valuations of these numbers? What does that suggest about the sequence in the 2-adic valuation?