Lab 1: Induction

Examples

Here are several examples to illustrate how you should write up an inductive proof. In general, I am more concerned with seeing that you have mastered the solution to a problem than in seeing a great deal of tedious, but obvious, detail in your answers. You should try to briefly write what you view as the key points that would neutralize the possible objections of any reasonable skeptic who is smart enough to see through the obvious details. Give high priority to avoiding saying anything that is incorrect.

- **Show that** \(1 + 2 + 3 + \ldots + n = n(n + 1)/2\).
  
  **Base case:** When \(n = 1\), the sum is 1, and \(n(n + 1)/2 = 1 * (2/2)\) is also 1.

  **Inductive step:** Suppose \(1 + 2 + \ldots + (n-1) = (n-1)n/2\) (that is, the claim is true all the way up to \(n-1\)). Then \(1 + 2 + \ldots + n = (1 + 2 + \ldots + (n-1)) + n = (n-1)n/2 + n = ((n-1)n + 2n)/2 = (n+1)n/2\), proving the claim is also true up to \(n\).

- **Show that** Mergesort is correct

  The procedure assumes the availability of procedure `Merge`, where \(\text{Merge } (A, p, q, r)\) sorts \(A[p, r]\) provided \(A[p, q]\) and \(A[q + 1, r]\) are both in sorted order.

  **Base case:** The procedure obviously works when \(p \geq r\), in which case the list has one element or is empty, hence already sorted.

  **Inductive step:** The procedure needs to sort a list \(A[p..r]\) of size \(n \geq 2\). Suppose it always works when the size of a list it needs to sort is less than \(n\). By this inductive hypothesis, \(A[p..q]\) and \(A[q+1..r]\), which are both of size less than \(n\), are put in sorted order by the two recursive calls. The required conditions are thus met for `Merge` to put \(A[p..r]\) in sorted order.

- **Show that you can always tile a** \(2^n \times 2^n\) **chessboard with one missing square, using L-shaped pieces, illustrated:**

  ![L-shaped pieces](image)

  **Base case:** It obviously works on a \(2^1 \times 2^1\) chessboard. (No need to write anything more to convince a skeptic here!)

  **Inductive step:** We want to show that it is possible to tile a \(2^n \times 2^n\) chessboard if it is possible to tile a \(2^{(n-1)} \times 2^{(n-1)}\) chessboard. So assume that it is possible to do the latter. Divide our \(2^n \times 2^n\) chessboard into four quadrants of size \(2^{(n-1)} \times 2^{(n-1)}\) each. This is possible since \(2^n\) is even. Lay a piece as shown in class. It remains to tile all but one square of each of the quadrants, which is possible if the inductive assumption holds.
Problems

1. Use induction to prove that
   \( \sum_{i=0}^{n-1} x^i = \frac{1-x^n}{1-x} \)
   \( \sum_{i=0}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

2. Use induction to show that the sum of the first \( k \) odd natural numbers is equal to \( k^2 \).

3. Prove that \( 3^n < n! \) for all \( n > 6 \).

4. Show that the number of ways to order (permute) the numbers from 1 to \( n \) is \( n! = 1 \cdot 2 \cdot 3 \cdot ... \cdot n \). Use induction.

5. A set of lines in a plane are in general position if no two of them are parallel, and no more than two of them intersect at any point. Such a set of lines divide the plane up into regions that have no lines running through them.
   
   (a) Show that there is always a way to color regions red and green so that no two regions that share a border are the same color.
   
   (b) Consider \( n > 2 \) lines in general position. Prove that at least one of the regions they form is a triangle.
   
   (c) Consider \( n > 2 \) lines in general position. Prove that these lines form at least \( n - 2 \) triangles.

6. Blue-eyed islanders: There is an island where there is a rule that if you ever find out that you have blue eyes then you must commit suicide that very night. Now, it just so happens that all of the \( n \) natives have blue eyes. However, no one knows they themselves have blue eyes and no one tells anyone else that the other has blue eyes. In fact, there is a very strict taboo against even mentioning eye color for reasons you will soon discover. As a result they all live in peace and no one commits suicide.

One day a missionary shows up on the island. Not intending harm, the missionary comments in front of the \( n \) assembled islanders that the ocean was as blue as the eyes of at least one of them. Now, the natives have exceptional abilities at logic so that on the \( n^{th} \) night there is a mass suicide.

   (a) Write down the inductive proof that shows why the missionary’s comments in front of \( n \) blue-eyed islanders causes a mass suicide on night \( n \).
   
   (b) Show if the missionary realizes his mistake after he utters his declaration, he can reduce the number of casualties by killing a blue-eyed person.
   
   (c) Show that if he agonizes too long about killing a blue-eyed person in part b, then he must kill two of them to save any lives.
7. What is wrong with this argument proving that all sheep are of the same color:

(a) Base case: One sheep. It is clearly the same color as itself.

(b) Inductive step: Given a flock of \( n \) sheep, take out sheep, \( a \). The remaining sheep must all be the same color by the induction hypothesis. Now put \( a \) back and pull out a different sheep, \( b \). By induction, the remaining sheep (containing \( a \)) are all the same color. Therefore \( a \) must be the same color as the rest of the sheep; hence all the sheep in the flock are of the same color.

8. Professors Johnson, Levenick, and Ruehr, have proposed the “elegant” sorting algorithm:

\[
\text{STOOGESORT}(A,i,j)
1 \quad \text{if } A[i] > A[j] \text{ then exchange } A[i] \text{ and } A[j]
2 \quad \text{if } i+1 \geq j \text{ then return}
3 \quad k = \lfloor (j-i+1)/3 \rfloor \quad \text{// Round down}
4 \quad \text{STOOGESORT}(A,i,j-k) \quad \text{// First two-thirds}
5 \quad \text{STOOGESORT}(A,i+k,j) \quad \text{// Last two-thirds}
6 \quad \text{STOOGESORT}(A,i,j-k) \quad \text{// First two-thirds again.}
\]

Using induction, show that stooge sort is correct.

9. We saw how to cover a chessboard that is missing one square with L-shaped tiles. On the Starship Enterprise, a version of chess is played on a triangular board where the spaces are also triangles. The players may choose how big a board they want to play on. The board must satisfy the following inductive definition of a valid board:

Base case: A triangle with no subdivisions is a size 1 chessboard.

Inductive step: A size 2\( n \) chessboard may be constructed from a size \( n \) chessboard by attaching four size \( n \) chessboards together. For example, here is a size 2 chessboard:

(a) Draw a picture of a size 8 chessboard.

(b) Give an inductive proof that the number of spaces on a triangular chessboard is always a power of 4.
(c) While waiting for Bones to make a move, Spock cuts off a triangular space from the corner of the chessboard, and wonders whether the remainder can be tiled with tiles that look like this:

That is, can a chessboard with one corner missing be tiled by pieces that look like a size 2 chessboard with one corner missing?

Show by induction that this can always be done.

10. The Romans sometimes practiced the punishment of *decimation* on their enemies, where they lined them up and killed every tenth one. (The word is related to the word *decimal*). No doubt, some overzealous Romans occasionally killed more than every tenth person ...

It is the first century A.D., and you are among a bunch of rebels that are captured by Roman soldiers. The soldiers want information about where to find your leader. You and the other rebels vow to guard that information to the death. The soldiers decide to line everybody up in a circle. Starting with person number 2, they will kill every other person clockwise round and round the circle until somebody chickens out and divulges the information, or until everybody is dead.

You decide to divulge. Your dilemma is that you don’t want your comrades in arms to witness your cowardice. So you decide to figure out where to stand in the circle so that you will be the last person standing, at which point you can abandon all pride, fall to your knees, and start groveling.

The problem: where should you stand?

It turns out that you can solve the problem recursively. Suppose there are an *even* number of people around the circle. Then look at who’s left by the time the Romans have made it around once:
This looks a lot like a smaller instance of the original problem:

(a) Suppose you could solve this smaller instance. How could you translate the solution into a position number that solves the original problem?

(b) Using your solution from part a, give pseudocode for recursive algorithm that solves the problem if the number of rebels is a power of 2. Prove that the algorithm works, and show that the proof doesn’t work when $n$ is not a power of 2.

(c) Generalize the pseudocode of part b so that it tells you where to stand, even if $n$ is not a power of 2.

(d) (You can answer this part, even if you didn’t get part c.) Show that if $n$ is a power of 2, you should always stand in position 1.

(e) Show that if $n = 2^m + k$, where $0 \leq k < 2^m$, you should stand in position $2k + 1$. This gives an easier way to solve the problem than running the algorithm of part d., though that algorithm can help you show this result.

(f) Show that you can solve the original problem for arbitrary $n$ by writing $n$ in binary, and moving the leading 1 to the end. For instance, if $n$ is 12, this is 1100 in binary, so you should stand in position 1001 = 9. (You can solve this part by assuming the claim of part e is correct, even if you didn’t succeed in showing it).

11. Find the greatest number that divides both 225277 and 178794 evenly.
   Hint: Begin by arguing that if a number divides both $a$ and $b$ evenly, then it divides $b$ and $a \mod b$ evenly. Then try to come up with pseudocode for an algorithm that works on arbitrary non-negative $a$ and $b$.

12. **Forty Thieves**, all of different ages, steal a huge pile of identical gold coins and must decide how to divide them up. They settle on the following procedure: the youngest divides them among the thieves however s/he wishes, then all 40 thieves vote on whether they are satisfied with the division. If a majority (21) vote NO, the youngest is killed, and the next youngest gets to try to divide the booty among the remaining 39 thieves (including him/herself), with the same penalty if a majority vote goes against the division. And so on.
Now, each of the 40 thieves is completely rational, a perfect logician, and knows that this is true of the others as well. Further, each thief always acts in her/his own self interest, ignoring (possibly) the interest of the group and plain old fairness. Given all this, how should the youngest of the 40 thieves divide the loot?

13. Programming problem: The Fibonacci function is defined as follows: \( \text{Fib}(1) = 1; \text{Fib}(2) = 1; \) and for all \( n > 2, \text{Fib}(n) = \text{Fib}(n-1) + \text{Fib}(n-2). \) Write a recursive Java function that computes it. Then add a (non-global) parameter that tells it whether to print anything as it computes it. If the parameter is set, then it should print out \( \text{Fib}(i) \) for each \( i \) from 1 to its parameter \( n \). Now for the tricky part: for each \( i \), it should print \( \text{Fib}(i) \) only once.