1. Let $X$ be a random variable such that

$$P(X = 1) = p = 1 - P(X = -1).$$

Find $c \neq 1$ such that $E[c^X] = 1$.

2. Let $X$ be a nonnegative, integer-valued random variable.

(a) Prove that

$$E[X] = \sum_{i=1}^{\infty} P(X \geq i).$$

Hint: For each $i \geq 1$, define

$$X_i = \begin{cases} 
1 & \text{if } X \geq i \\
0 & \text{if } X < i 
\end{cases}$$

Express $X$ in terms of $X_i$'s.

(b) We say that $X$ is stochastically larger than $Y$, written $X \geq_{st} Y$, if for all $t$,

$$P(X \geq t) \geq P(Y \geq t).$$

Use part (a) to prove that if $X \geq_{st} Y$, then $E[X] \geq E[Y]$.

3. Determine the value of $a$ that minimizes $E[(X - a)^2]$ for a random variable $X$.

4. (a) Prove the Cauchy-Schwarz inequality, namely, that for random variables $X$ and $Y$,

$$(E[XY])^2 \leq E[X^2]E[Y^2].$$

Hint: Use the fact that for all $t$, $(X - tY)^2 \geq 0$.

(b) Use part (a) to prove an application of Jensen’s Inequality, namely, that

$$(E[X])^2 \leq E[X^2].$$