1. Prove that if \( \lim(x_n) = x \) and if \( x > 0 \), then there exists a natural number \( M \) such that \( x_n > 0 \) for all \( n \geq M \).

Proof. (Greg Henselman) For all \( \varepsilon > 0 \), there exists \( m \in \mathbb{N} \) such that for all \( n \geq m \), \( 0 \leq |x - x_n| < \varepsilon \). Let \( \varepsilon = x \). Thus we have

\[
|x - x_n| < x \iff -x < x - x_n < x \iff 0 < x_n < 2x.
\]

Combining the last inequality with the first statement of this proof, we have it that there exists \( m \in \mathbb{N} \) such that for all \( n \geq m \), \( x_n > 0 \).

2. Show that \( \lim \left( \frac{n^2}{n!} \right) = 0 \).

Proof. (Jessica Fujii) \( n^2/n! = n/(n - 1)! \). Ratio Test:

\[
\frac{n+1}{n} = \frac{n+1}{n!} \cdot \frac{(n-1)!}{n} = \frac{n+1}{n^2}.
\]

\[
\lim(1/n) + \lim(1/n^2) = 0.
\]

Therefore, \( (x_n) \) is convergent and \( \lim(n^2/n!) = 0 \).

3. Prove that if \( \lim(x_n) = 0 \) and \( (y_n) \) is a bounded sequence (but not necessarily convergent), then \( \lim(x_ny_n) = 0 \).

Proof. (Daniel Gossard) Let \( \varepsilon > 0 \). By the definition of a convergent sequence, we know that \( \lim(x_n) = 0 \) implies that there exists \( K \in \mathbb{N} \) such that for all \( n \geq K \), \( |x_n| < \varepsilon/M \). Here, \( M \) is the bound on \( (y_n) \) according to the definition. We also know that \( |y_n| \leq M \) for all \( n \in \mathbb{N} \).

Now, we can see that for all \( n \geq K \),

\[
|x_n y_n| = |x_n||y_n| \leq |x_n|M < \frac{\varepsilon}{M}M = \varepsilon.
\]

Therefore, since \( \varepsilon \) is always greater than 0, we see that \( \lim(x_ny_n) = 0 \).
4. Let $X = (x_n)$ be a sequence of positive real numbers such that $\lim \left( \frac{x_{n+1}}{x_n} \right) = L > 1$. Show that $X$ is not a bounded sequence and hence is not convergent.

**Proof.** (Steve Lester) Let $(x_n)$ be a sequence of positive real numbers such that $\lim(x_{n+1}/x_n) > 1$. Define $L := \lim(x_{n+1}/x_n)$. Let $r \in \mathbb{R}$ such that $L > r > 1$. Define $\varepsilon := L - r > 0$. By definition, there exists $K \in \mathbb{N}$ such that for all $n \geq K$

$$L - \varepsilon < \frac{x_{n+1}}{x_n} \quad \text{so} \quad (L - \varepsilon)x_n < x_{n+1}.$$ Continue applying this relation to get

$$(L - \varepsilon)x_n > (L - \varepsilon)^2x_{n-1} > \cdots > (L - \varepsilon)^{n-K}x_K \Rightarrow rx_n > r^{n-K}x_K \Rightarrow x_n > r^{n-K}x_K.$$ Let $c := x_K/r^K$. Note that $c$ is constant. So $x_n > r^n c$ for all $n \geq K$. But $r > 1$ so $(cr^n)$ tends to $+\infty$. Thus $(x_n)$ tends to $+\infty$ and is unbounded. \hfill \qed

5. Suppose that $X = (x_n)$ is a convergent sequence and $Y = (y_n)$ is another sequence such that for any $\varepsilon > 0$ there exists $M$ such that $|x_n - y_n| < \varepsilon$ for all $n \geq M$. Prove that $Y$ converges and $\lim X = \lim Y$.

**Proof.** (Paige Cudworth) Since $(x_n)$ is convergent, there exists $x \in \mathbb{R}$ such that for $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x - x_n| < \varepsilon/2$ for all $n \geq N$. So if we let $p := \max\{N, M\}$, we have $|x - x_n| < \varepsilon/2$ and $|x_n - y_n| < \varepsilon/2$ for all $n \geq p$. Using the Triangle Inequality, we get

$$|x - y_n| = |x - x_n + x_n - y_n| \leq |x - x_n| + |x_n + y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ So we have $|x - y_n| < \varepsilon$ for all $n \geq p$. So $x$ is the limit of $(y_n)$. \hfill \qed

6. Let $x_1 \geq 2$ and $x_{n+1} := 1 + \sqrt{n-1}$ for $n \in \mathbb{N}$. Prove that the sequence $(x_n)$ is convergent and find the limit.

**Proof.** (Elizabeth Alford) Using induction: For $n = 1$, $x_1 \geq 2$ by assumption. Then inductively, we assume that $x_k \geq 2$. We want to show that $x_{k+1} \geq 2$.

$$x_{k+1} \geq 2 \iff 1 + \sqrt{x_k - 1} \geq 2 \iff x_k - 1 \geq 1 \iff x_k \geq 2.$$ Therefore, $x_n \geq 2$ for all $n \in \mathbb{N}$. Then we need to show that $(x_n)$ is decreasing.

$$x_{k+1} \leq x_k \iff 1 + \sqrt{x_k - 1} \leq x_k \iff 0 \leq (x_k - 1)(x_k - 2).$$ Since $x_k \geq 2$ for all $k \in \mathbb{N}$, the last inequality holds and thus $(x_n)$ is decreasing. Therefore, by MCT, $x := \lim(x_n)$ exists. Then to find what this limit is:

$$x := \lim(x_n) = \lim(x_{n+1})$$ by tail-sequences. We then can solve for $x$ to find our limit.

$$x = 1 + \sqrt{x - 1} \iff x^2 - 3x + 2 = 0 \iff (x - 1)(x - 2) = 0.$$ Thus $x = 1$ or $x = 2$. But since $x_n \geq 2$ for all $n \in \mathbb{N}$, $x = 2$. \hfill \qed
7. Let \((x_n)\) be a bounded sequence and for each \(n \in \mathbb{N}\) let \(s_n := \sup\{x_k : k \geq n\}\) and \(t_n := \inf\{x_k : k \geq n\}\). Prove that \((s_n)\) and \((t_n)\) are convergent sequences and prove that if \(\lim(s_n) = \lim(t_n)\), then \((x_n)\) is convergent. (One calls \(\lim(s_n)\) the \textbf{limit superior} of \((x_n)\) and \(\lim(t_n)\) the \textbf{limit inferior} of \((x_n)\).)

\textit{Proof.} (Andrea Walker) Let \(A_m := \{x_k : k \geq M\}\). Then \(A_{m+1} \subset A_m\) and thus

\[
\sup A_{m+1} \leq \sup A_m \quad \text{and} \quad \inf A_{m+1} \geq \inf A_m.
\]

Therefore, \((s_n)\) is a decreasing sequence and \((t_n)\) is an increasing sequence. We know that

\[
t_1 \leq t_n \leq s_n \leq s_1
\]

for all \(n \in \mathbb{N}\) and thus \(t_1\) is a lower bound for \((s_n)\) and \(s_1\) is an upper bound for \((t_n)\). Therefore, by the MCT, \((s_n)\) and \((t_n)\) are convergent sequences.

Suppose \(\lim(s_n) = \lim(t_n)\). By definition of \(s_n\) and \(t_n\), we have

\[
t_n \leq x_n \leq s_n
\]

for all \(n \in \mathbb{N}\). By the Squeeze Theorem, \(\lim(t_n) = \lim(x_n) = \lim(s_n)\). \(\square\)